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The Small Scales of the Stochastic Navier–Stokes Equations Under Rough Forcing

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We prove that the small scale structures of the stochastically forced Navier–Stokes equations approach those of the naturally associated Ornstein–Uhlenbeck process as the scales get smaller. Precisely, we prove that the rescaled kth spatial Fourier mode converges weakly on path space to an associated Ornstein–Uhlenbeck process as $|k| \to \infty$. In addition, we prove that the Navier–Stokes equations and the naturally associated Ornstein–Uhlenbeck process induce equivalent transition densities if the viscosity is replaced with hyperviscosity. This gives a simple proof of unique ergodicity for the hyperviscous Navier–Stokes system. We show how different strengthened hyperviscosity produce varying levels of equivalence.

1. INTRODUCTION

We consider the stochastically forced Navier–Stokes equations

$$\begin{aligned} &\frac{\partial \omega}{\partial t}(x,t) + (u(x,t)\cdot\nabla)\omega(x,t) = \nu\Delta\omega(x,t) + \frac{\partial W}{\partial t}(x,t), \\ &\omega(x,0) = \omega_0(x), \end{aligned} \tag{1}$$

on the two dimensional 2π-periodic domain, $\mathbf{T}^2$. The velocity, $u(x,t)$, is recovered from the vorticity, $\omega(x,t)$, by the Biot–Savart law (see for instance ref. 10). We assume that the fluid has no mean flow: $\int_{\mathbf{T}^2} u(x,t)dx = 0$. The stochastic forcing is generated by a Brownian motion of the form $W(x,t) = \sum_{k\in\mathbb{Z}^2_*} \sigma_k \exp(ix\cdot k)\beta_k(t)$, where $\mathbb{Z}^2_*$ denotes $\mathbb{Z}^2/\{0\}$,

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the $\sigma_k \in \mathbb{C}$ are non-zero complex coefficients satisfying $\sigma_k = \overline{\sigma_{-k}}$, and the β_k are identically distributed standard complex Brownian motions, which are mutually independent except for the condition $\beta_k = \overline{\beta_{-k}}$. Setting $\omega(x,t) = \sum_{k\in\mathbb{Z}^2_*} \omega_k(t)\exp(ik\cdot x)$, Eq. (1) becomes a collection of coupled Itô differential equations:

$$d\omega_k(t) = \Big[-\nu|k|^2\omega_k(t) + 2i \sum_{j+\ell=k} \frac{\ell^\perp \cdot k}{|\ell|^2}\omega_\ell(t)\omega_j(t)\Big]dt + \sigma_k\, d\beta_k(t). \qquad (2)$$

Some of our results will only hold for a modified version of (2) where the effect of the viscosity has been enhanced by adding 'hyper-viscosity'. For any $\alpha > 0$, we consider the system of equations

$$d\omega_k(t) = \Big[-\nu|k|^\alpha\omega_k(t) + 2i \sum_{j+\ell=k} \frac{\ell^\perp \cdot k}{|\ell|^2}\omega_\ell(t)\omega_j(t)\Big]dt + \sigma_k\, d\beta_k(t). \qquad (3)$$

This is the Fourier representation of a partial differential equation of the form (1), where $\nu\Delta$ has been replaced by $\nu\Delta^{\alpha/2}$.

Consider the Ornstein–Uhlenbeck process

$$\begin{aligned} &\frac{\partial z}{\partial t}(x,t) = \nu\Delta^{\frac{\alpha}{2}} z(x,t) + \frac{\partial W}{\partial t}(x,t), \\ &z(x,0) = \omega_0(x). \end{aligned} \qquad (4)$$

If $z(x,t) = \sum_{k\in\mathbb{Z}^2_*} z_k(t)\exp(ik\cdot x)$, then (4) becomes

$$dz_k(t) = -\nu|k|^\alpha z_k(t)dt + \sigma_k\, d\beta_k(t)\ . \qquad (5)$$

This Ornstein–Uhlenbeck process is the natural linear PDE associated with the stochastic Navier–Stokes equations (SNS) (3). Henceforth, we assume that for $k \neq 0$,

$$\frac{K_1}{|k|^l} \leqslant |\sigma_k| \leqslant \frac{K_2}{|k|^l}, \qquad (6)$$

for some positive constants K_1, K_2, and l. We assume $\sigma_0 = 0$ in order to ensure that there is no mean flow.

In this note we give partial answers to the following questions. When are the fine scale structures of (1) the same as those of (4)? Can ω_k be viewed as a perturbation of z_k when $|k|$ is large enough? A lack of precise understanding of the small scale structure is one of the major technical impediments to a straightforward, Markovian analysis of many stochastic partial differential equations. See ref. 9 for a discussion of the relationship between the small scale structures and some approaches to proving ergodicity.

The three theorems in this section offer answers to different aspects of these questions. The first theorem demonstrates the weak convergence of the small scales of (1) to those of (4). The second theorem characterizes the relationship of the small scales in terms of the equivalence of the Markov transition densities. The third theorem characterizes this relationship in terms of the equivalence on the entire path space of the dynamics.

Let

$$\omega'_k = \frac{\sqrt{2}|k|^{\alpha/2}}{|\sigma_k|}\omega_k \quad \text{and} \quad z'_k = \frac{\sqrt{2}|k|^{\alpha/2}}{|\sigma_k|}z_k.$$

With this rescaling, z'_k is a complex Ornstein–Uhlenbeck process with mean zero and variance one for any k. First, we show that as $|k_1|, \dots, |k_d| \to \infty$, $(\omega'_{k_1}, \dots, \omega'_{k_d})$ converges to $(z'_{k_1}, \dots, z'_{k_d})$ on any finite time interval in a sense made precise below. By $|k_1|, \dots, |k_d| \to \infty$, we will always mean $\min_{i\in\{1,\dots,d\}} |k_i| \to \infty$.

Theorem 1. In the above setting, the following two convergence results hold for any fixed finite $t > 0$:

(i) Assume that the initial conditions satisfy: $|\omega_k(0)| < \frac{\mathcal{D}}{|k|^r}$ for some $\mathcal{D}, r > 0$, such that $\limsup_{k\to\infty} |\sigma_k|^2 |k|^{2r-\alpha} < 2$. Then, for any bounded uniformly continuous function $G: C([0,t]; \mathbb{C}^d) \to \mathbb{R}$,

$$E|G(\omega'_{k_1}, \dots, \omega'_{k_d}) - G(z'_{k_1}, \dots, z'_{k_d})| \to 0,$$

as $|k_1|, \dots, |k_d| \to \infty$, where E denotes the expectation with respect to the driving Brownian motions.

(ii) For any continuous bounded function $G: C([0,t]; \mathbb{C}^d) \to \mathbb{R}$,

$$E_{\mu^z} E|G(\omega'_{k_1}, \dots, \omega'_{k_d}) - G(z'_{k_1}, \dots, z'_{k_d})| \to 0,$$

as $|k_1|, \dots, |k_d| \to \infty$. Here, E_{μ^z} denotes the expectation with respect to the initial conditions, the distribution of which is given by the stationary measure of the unscaled z process.

While this is already an interesting statement, one might like strong analytic control rather than weak convergence. In refs. 4 and 6, the Bismut–Elworthy–Li formula was used to prove the absolute continuity of the time t transition densities of the SNS starting from different initial conditions. This, in turn, was used to prove a delicate ergodic theorem. In order to apply the Bismut–Elworthy–Li formula, precise knowledge of the spatial regularity of the SNS was needed. Their technique made use of the fact that the $z(t)$ is less regular (in space) than $u(t) - z(t)$; hence, the spatial regularity of $u(t)$ is determined by that of $z(t)$. In light of this, one might hope to prove the stronger statement that the distribution of $\{\omega_k\}_{k\in\mathbb{Z}^2_*}$ is absolutely continuous with respect to that of $\{z_k\}_{k\in\mathbb{Z}^2_*}$. One could think of this holding either on path space or at a moment of time, t. For any $\alpha > 2$, we prove equivalence of the respective transition densities. For $\alpha > 4$, we prove equivalence on path space. Precisely,

Theorem 2. For $\alpha > 2$ and $t > 0$, the measures induced on $l^2(\mathbb{Z}^2_*)$ by $z(t)$ and $\omega(t)$ are mutually absolutely continuous if $z(0) = \omega(0)$.

Theorem 3. For $\alpha > 4$ and $t > 0$, the measures induced on $C([0,t]; l^2(\mathbb{Z}^2_*))$ by z and ω are mutually absolutely continuous if $z(0) = \omega(0)$.

Here, $l^2(\mathbb{Z}^2_*)$ is the space of square summable sequences of complex numbers indexed by $\mathbb{Z}^2_*$. In Section 5.3, we use this result to prove that the hyperviscous SNS system is uniquely ergodic.

Corollary 1.1. If $\alpha > 2$, then equation (2) has a unique invariant measure and this measure is equivalent to the unique invariant measure of Eq. (5).

This result could likely be proved using the methods in refs. 4 and 6; in fact, it is weaker than the ergodic results in these papers since it requires slight hyperviscosity. However, this note gives the equivalence of the invariant measure of Eq. (3) with respect to the invariant measure of Eq. (5). More importantly, the method we present gives different intuition about why the system is ergodic. There are also methods to prove Eq. (1) is ergodic by using estimates which are fundamentally non-Markovian; see for example refs. 1, 3, 7, and 9. The last reference contains an overview of these less standard techniques. Here, we stay in the Markovian framework. The analysis in this paper can be carried out for the Burgers equation without any hyperviscosity as the sums in the Girsanov term are one dimensional.

This paper is organized as follows. In Section 2, we make several deterministic observations about solutions to the SNS process and the associated Ornstein–Uhlenbeck process. In Section 3, we estimate the probabilities of the deterministic picture suggested in Section 2 and show that this picture is correct with high probability. In Section 4, we prove the small scale limit theorem. In Section 5, we prove the unique ergodicity of the hyperviscous Navier–Stokes equations by proving that the SNS and the Ornstein–Uhlenbeck processes are absolutely continuous on path space for $\alpha > 4$ and have absolutely continuous time t marginals at any fixed time t if $\alpha > 2$.

2. DETERMINISTIC OBSERVATIONS

We define the following useful norms and subsets of path space. Let $\omega = \{\omega_k\}_{k \in \mathbb{Z}^2_*} \in l^2(\mathbb{Z}^2_*)$. Define

$$|\omega|_{\infty,\gamma} = \sup_{k \in \mathbb{Z}^2_*} |k|^\gamma |\omega_k| \quad \text{and} \quad \|\omega\| = \left(\sum |\omega_k|^2\right)^{1/2},$$

and the subsets of path space

$$A^t_1(\mathcal{D}, \gamma) = \left\{z \in C([0,t], l^2(\mathbb{Z}^2_*)) : |z(s)|_{\infty,\gamma} \leqslant \mathcal{D}, \forall s \in [0,t]\right\},$$
$$A^t_2(\mathcal{E}, \eta) = \left\{\omega \in C([0,t], l^2(\mathbb{Z}^2_*)) : \|\omega(0)\|^2 \leqslant \mathcal{E}, \|\omega(s)\|^2 \leqslant \eta\mathcal{E}, \forall s \in [0,t]\right\}.$$

The arguments in this section are related to those in ref. 11. We define 'trapping' regions, along whose boundary the vector field corresponding to the dynamics points inward; hence, solutions are trapped within these regions for all time.

Let $\alpha' \in (1, \alpha]$, and define $K_0(\gamma, \mathcal{E}, \eta, \alpha')$ to be the smallest integer such that

$$\left(C(\gamma)\sqrt{\eta\mathcal{E}}\frac{|k|\sqrt{\log|k|}}{\nu|k|^{\alpha'}}\right) < \frac{1}{6},$$

for all k such that $|k| > K_0$. $C(\gamma)$ is a constant which only depends on γ through summation formulas; it is made explicit in the appendix.

It will be useful to set $\mathcal{D}' = 2\sqrt{\eta\mathcal{E}}K_0^\gamma$. This constant is picked to ensure that the enstrophy, $\sqrt{\eta\mathcal{E}}$, helps control some $|\cdot|_{\infty,\gamma}$ norm, once K_0 is determined.

As stated before, we are interested in comparing solutions of the Ito stochastic differential equations given in (3) and (5). To accomplish this, we study the difference of these two processes. Let $\rho = \omega - z$ and $\rho(0) = 0$. $\rho = \{\rho_k\}_{k\in\mathbb{Z}^2_*}$ satisfies the system of random ordinary differential equations:

$$\frac{d\rho_k(t)}{dt} = -\nu|k|^\alpha \rho_k(t) + F(\omega)_k(t), \tag{7}$$

where $F(\omega)_k(t)$ is the nonlinear term in the drift of (3). Proposition 2.1 gives sufficient control of $\rho_k(t)$.

Proposition 2.1. If $z \in A^t_1(\mathcal{D}, \gamma)$ and $\omega \in A^t_2(\mathcal{E}, \eta)$, then

$$\sup_{s\in[0,t]} |\rho_k(s)| \leqslant \frac{2\bar{\mathcal{D}}}{|k|^{\gamma+(\alpha-\alpha')}}$$

for all k with $|k| > K_0(\gamma, \mathcal{E}, \eta, \alpha')$, where $\bar{\mathcal{D}} = 2\max\{\mathcal{D}, \mathcal{D}'\}$. Hence, $\omega \in A^t_1(3\bar{\mathcal{D}}, \gamma) \cap A^t_2(\mathcal{E}, \eta)$.

Before proving this proposition, we state the Technical Lemma 2.2 proved in Appendix A.

Lemma 2.2. If $\omega \in A^t_1(\mathcal{D}, \gamma) \cap A^t_2(\mathcal{E}, \eta)$, then

$$\sup_{s\in[0,t]} |F(\omega)_k(s)| \leqslant (C(\gamma)\sqrt{\eta\mathcal{E}}|k|\sqrt{\log|k|})\frac{\bar{\mathcal{D}}}{|k|^\gamma}.$$

Proof of Proposition 2.1. We begin by noting that for k with $|k| \leqslant K_0(\gamma, \mathcal{E}, \eta, \alpha')$, the estimate $|\omega_k(s)| \leqslant \frac{\mathcal{D}'}{|k|^\gamma}$ holds for all $s \in [0, t]$; this is because $\omega \in A^t_2(\mathcal{E}, \eta)$, and $\mathcal{D}'$ has been chosen so that this estimate holds.

Suppose that for some $s \in [0, t]$ and some k with $|k| > K_0$,

$$|\rho_k(s)| = \frac{2\bar{\mathcal{D}}}{|k|^{\gamma+(\alpha-\alpha')}}.$$

Suppose that for all k' with $|k'| > K_0$ and $|k'| \neq k$,

$$|\rho_{k'}(s)| \leqslant \frac{2\bar{\mathcal{D}}}{|k'|^{\gamma+(\alpha-\alpha')}}.$$

We show that the vector field points inward at this point; hence, $\{\rho_k\}$ cannot violate the inequality in Proposition 2.1. By assumption on z at time s,

$$|\omega_k(s)| \leqslant \frac{\mathcal{D}}{|k|^\gamma} + \frac{2\bar{\mathcal{D}}}{|k|^{\gamma+(\alpha-\alpha')}} \leqslant \frac{3\bar{\mathcal{D}}}{|k|^\gamma}. \tag{8}$$

By the Lemma 2.2,

$$|F(\omega)_k(s)| \leqslant \left(C(\gamma)\sqrt{\eta\mathcal{E}}|k|\sqrt{\log|k|}\right)\frac{6\bar{\mathcal{D}}}{|k|^\gamma}. \tag{9}$$

Computing $|k|^\alpha|\rho_k(s)| = \frac{2\bar{\mathcal{D}}}{|k|^\gamma}|k|^{\alpha'}$, and using the fact that $|k| > K_0$, we have

$$\nu|k|^\alpha|\rho_k(s)| > |F(\omega)_k(s)|.$$

Multiplying the equation for ρ_k by $\bar{\rho}_k$ produces

$$\begin{aligned}\frac{1}{2}\frac{d}{ds}|\rho_k(s)|^2 &= -\nu|k|^\alpha|\rho_k(s)|^2 + F(\omega)_k(s)\bar{\rho}_k(s)\\ &\leqslant \left(-\nu|k|^\alpha|\rho_k(s)| + |F(\omega)_k(s)|\right)|\rho_k(s)|.\end{aligned}$$

This implies that $|\rho_k(s)|$ must decrease at time s, since the vector field of the random ODE for ρ_k points inward. ▮

3. PROBABILISTIC ESTIMATES

In this section, we show that for certain choices of their defining parameters, the events or sets in path space defined in the previous section occur with high probability. The following two lemmas give conditions under which these events occur with high probability.

Lemma 3.1. Fix a $\mathcal{E} > 0$ and finite $t > 0$. For any SNS initial condition, $\omega(0)$, satisfying $\|\omega(0)\|^2 \leqslant \mathcal{E}$,

$$P\left(\omega \in A_2^t(\mathcal{E},\eta)\right) \geqslant 1 - \exp\left\{-\frac{\nu}{\sigma_{\max}^2}[(\eta-1)\mathcal{E} - \mathcal{E}_1 t]\right\}, \tag{10}$$

for η sufficiently large, $\sigma_{\max} = \max_{k\in\mathbb{Z}_*^2}|\sigma_k|$, and $\mathcal{E}_1 = \sum_{k\in\mathbb{Z}_*^2}|\sigma_k|^2$.

Proof. The proof of an almost identical result can be found in ref. 8. By Itô's formula, we have

$$\begin{aligned}\|\omega(t)\|^2 &= \|\omega(0)\|^2 + \mathcal{E}_1 t + 2\int_0^t \sum_k \sigma_k \cdot \omega_k(s) dB_k(s) - 2\nu \int_0^t \sum_k |k|^\alpha |\omega_k(s)|^2 ds \\ &= \|\omega(0)\|^2 + \mathcal{E}_1 t + N_t,\end{aligned}$$

where $N_t \leqslant M_t - \frac{1}{2}\frac{\nu}{\sigma_{\max}^2}\langle M\rangle_t$, $M_t = 2\sum_k \int_0^t \sigma_k \cdot \omega_k(s) dB_k(s)$, and $\langle M\rangle_t$ is the quadratic variation of the martingale M_t. The standard exponential martingale estimate for L^2-martingales gives

$$P\left(\sup_{s\in[0,t]} N_s > (\eta-1)\mathcal{E} - \mathcal{E}_1 t\right) \leqslant \exp\left\{-\frac{\nu}{\sigma_{\max}^2}[(\eta-1)\mathcal{E} - \mathcal{E}_1 t]\right\},$$

for η sufficiently large; this is the desired estimate. ∎

We now state a simple lemma for the Ornstein–Uhlenbeck process.

Lemma 3.2. Fix $r > 0$ so that $\limsup_{k\to\infty} |\sigma_k|^2 |k|^{2r-\alpha} < 2$. If $|z(0)|_{\infty,r} \leqslant \infty$, then

$$P\left(z \in A_1^t(\mathcal{D}, r)\right) \to 1, \tag{11}$$

for any fixed $t > 0$ as $\mathcal{D} \to \infty$.

Under the conditions of the above lemmas: if $z(0) = \omega(0)$, $|z(0)|_{\infty,\gamma} < \infty$, and $\|\omega(0)\| \leqslant \mathcal{E}$, then for any fixed $\delta > 0$, we can find $\mathcal{D}$ and η so that

$$P\left(z \in A_1^t(\mathcal{D}, r),\ \ \omega \in A_2^t(\mathcal{E}, \eta)\right) > 1 - \delta. \tag{12}$$

Combining these lemmas with Proposition 2.1, we find that with probability at least $1-\delta$,

$$\sup_{s\in[0,t]} |\rho_k(s)| \leqslant \frac{2\bar{\mathcal{D}}}{|k|^{r+(\alpha-\alpha')}},$$

and

$$P\left(\omega \in A_1^t(3\bar{\mathcal{D}}, r) \cap A_2^t(\mathcal{E}, \eta)\right) > 1 - \delta.$$

4. LIMIT THEOREM

In this section, we prove Theorem 1, which states that the high modes of the stochastic Navier–Stokes system, when scaled appropriately, converge in $C([0,t];\mathbb{C}^d)$ to a standard complex Ornstein–Uhlenbeck process. We use the pathwise control gained in Sections 1–3 to prove this.

Let $G: C([0,t],\mathbb{C}^d)\to\mathbb{R}$ be a bounded uniformly continuous function. Fix $\epsilon>0$. Pick $\delta>0$ so that if $\|x-y\|_\infty<\delta$, then $|G(x)-G(y)|<\epsilon$. Here, $\|\cdot\|_\infty$ denotes the sup norm in time. Define

$$z'_k=\frac{\sqrt{2}|k|^{\frac{\alpha}{2}}}{|\sigma_k|}z_k,\qquad \omega'_k=\frac{\sqrt{2}|k|^{\frac{\alpha}{2}}}{|\sigma_k|}\omega_k,\qquad \text{and}\qquad \rho'_k=\omega'_k-z'_k.$$

Note that z'_k is a standard complex Ornstein–Uhlenbeck process with mean 0 and variance 1. Fix $\epsilon'>0$. By the estimates of Sections 2 and 3 and the assumption on the initial condition, we know that there exists a constant, K, such that $P(\|\rho'_{k_1}\|_\infty+\cdots+\|\rho'_{k_d}\|_\infty<\delta)>1-\epsilon'$ for all $|k_i|\geqslant K$. This gives

$$E\big|G(\omega'_{k_1},\dots,\omega'_{k_d})-G(z'_{k_1},\dots z'_{k_d})\big|\leqslant\epsilon P\big(\|\rho'_{k_1}\|_\infty+\cdots+\|\rho'_{k_d}\|_\infty<\delta\big)+2\bar{G}\epsilon'.$$

Here, $\bar{G}=\sup_x|G(x)|$. Since ϵ,ϵ' are arbitrary, we have proven the first part of Theorem 1.

Now let $G: C([0,t];\mathbb{C}^d)\to\mathbb{R}$ be a bounded continuous function. Fix $\epsilon>0$. Let δ_n be a sequence of positive numbers tending to 0 and define

$$A_n=\{x\in C([0,t];\mathbb{C}^d):\|x-y\|_\infty<\delta_n\Rightarrow|G(x)-G(y)|<\epsilon\}.$$

Since G is a continuous function, $\bigcup_{n=1}^\infty A_n=C([0,t];\mathbb{C}^d)$. Setting $k_*=\min_i|k_i|$, the definition of A_n implies that

$$\begin{aligned}E_{\mu^z}E\big|G(\omega'_{k_1},\dots,\omega'_{k_d})-G(z'_{k_1},\dots,z'_{k_d})\big|&\leqslant 2\bar{G}P\big((z'_{k_1},\dots,z'_{k_d})\in A^c_{k_*}\big)\\&+\epsilon P\big(\|(\omega'_{k_1},\dots,\omega'_{k_d})-(z'_{k_1},\dots,z'_{k_d})\|_\infty<\delta_{k_*};(z'_{k_1},\dots,z'_{k_d})\in A_{k_*}\big)\\&+2\bar{G}P\big((z'_{k_1},\dots,z'_{k_d})\in A_{k_*};\|(\omega'_{k_1},\dots,\omega'_{k_d})-(z'_{k_1},\dots,z'_{k_d})\|_\infty\geqslant\delta_{k_*}\big).\end{aligned}\tag{13}$$

Notice that

$$\mu^z((z'_{k_1},\dots,z'_{k_d})\in A_{k_*})\to 1\ \text{as}\ |k_1|,\dots,|k_d|\to\infty,$$

since for all k the z'_k are identically distributed and $A_{k_*} \to C([0,t];\mathbb{C}^d)$ as $|k_1|,\ldots,|k_d| \to \infty$. This estimate insures that the first term in (13) goes to zero as $|k_1|,\ldots,|k_d| \to \infty$.

By combining the estimate from (12) and Proposition 2.1 at the end of Section 6, we see that if $\delta_n \to 0$ sufficiently slowly then

$$P\Big(\|(\omega'_{k_1},\ldots,\omega'_{k_d}) - (z'_{k_1},\ldots,z'_{k_d})\|_\infty \geqslant \delta_{k_*}\Big) \to 0 \quad \text{as } |k_1|,\ldots,|k_d| \to \infty.$$

Thus, the third term in (13) goes to zero. Since the second term is bounded by ϵ, which was arbitrary, we have proven the second statement of the theorem. ∎

This shows that $(\omega'_{k_1},\ldots,\omega'_{k_d})$ approaches a standard d-dimensional complex Ornstein–Uhlenbeck process in distribution as $|k_1|,\ldots,|k_d| \to \infty$. We remark that for any fixed indices $k_1,\ldots,k_d$, Girsanov's theorem establishes equivalence of the measures on path space. But, the estimates on the Novikov term worsen as the indices tend to ∞. This is because the estimates on the non-linearity (in Sections 1–3) grow in $|k|$.

5. ERGODICITY AND ABSOLUTE CONTINUITY

5.1. Absolute Continuity of ω and z in Path Space when $\alpha > 4$

In this subsection, we show that the measures induced by ω and z on $C([0,t], l^2(\mathbb{Z}^2_*))$ are equivalent (mutually absolutely continuous) if $z(0) = \omega(0)$ and $\alpha > 4$. We appeal to Girsanov's theorem through Lemma B.1.

The equations governing $z(t)$ and $\omega(t)$ differ by the nonlinear term. To apply Lemma B.1, we need to show that

$$\int_0^t \sum_{k\in\mathbb{Z}^2_*} \frac{|F_k(\omega(s))|^2}{|\sigma_k|^2} \mathbf{1}(\mathcal{B})ds < C(\mathcal{B}) < \infty$$

for some measurable choice of $\mathcal{B} \subset C([0,t], l^2(\mathbb{Z}^2_*))$ and some constant C, which might depend on $\mathcal{B}$. Then, the Lemma implies that the measures on path space are equivalent when restricted to $\mathcal{B}$. If for any $\delta > 0$, one can find such a $\mathcal{B}$, satisfying $P\{\omega \in \mathcal{B}\} > 1 - \delta$, then Theorem 3 follows from Lemma B.1.

Given any $\delta, \epsilon > 0$, there exists a constant, $\mathcal{D}$, such that

$$P\left\{z \in A_1^t\left(\mathcal{D}, \frac{\alpha}{2} + l - \epsilon\right)\right\} > 1 - \delta.$$

Hence, by Proposition 2.1, there exist a constants, $\mathcal{E}, \eta$, so that

$$P\left\{\omega \in A_1^t(3\bar{\mathcal{D}}, \frac{\alpha}{2}+l-\epsilon) \cap A_2^t(\mathcal{E}, \eta)\right\} > 1-\delta.$$

Set $\mathcal{B} = A_1^t(3\bar{\mathcal{D}}, \frac{\alpha}{2}+l-\epsilon) \cap A_2^t(\mathcal{E}, \eta)$. For $\omega \in \mathcal{B}$, by Lemma 2.2, one has

$$\sup_{s\in[0,t]} \sum_{k\in\mathbb{Z}_*^2} \frac{|F_k(\omega(s))|^2}{|\sigma_k|^2} \leqslant C(K_2, \bar{\mathcal{D}}, \mathcal{E}, \eta) \sum_{k\in\mathbb{Z}_*^2} \frac{|k|^{2l}|k|^2 \log|k|}{|k|^{\alpha+2l-2\epsilon}}.$$

Since ϵ is an arbitrary positive number, this sum is finite if $\alpha > 4$, proving the result.

5.2. Absolute Continuity of Time t Marginals of ω and z when $\alpha > 2$

We show that if $\alpha > 2$ and t is fixed the distributions of the $l^2(\mathbb{Z}_*^2)$-valued random variables $\omega(t)$ and $z(t)$ are mutually absolutely continuous. We use a technique, which is inspired by a variation on the Bismut–Elworthy–Li formula developed by the first author in collaboration with Martin Hairer. In order to do this, we need Lemma C.1, which controls convergence of densities given uniform control of associated relative entropies. Fixing the terminal time t, it is sufficient to construct an auxiliary stochastic process $\tilde{\omega}(s)$ such that $\omega(t) = \tilde{\omega}(t)$ and $\tilde{\omega}$ is equivalent to z on the path space $C([0,t], l^2(\mathbb{Z}_*^2))$.

Setting

$$\tilde{F}_k(s) = \begin{cases} 0, & s < \frac{t}{2} \text{ or } s > t, \\ 2e^{-\nu|k|^\alpha(t-s)} F_k(\omega(2s-t)), & s \in [\frac{t}{2}, t], \end{cases}$$

we define $\tilde{w}$ by

$$\begin{aligned} d\tilde{\omega}_k(s) &= [-\nu|k|^\alpha \tilde{w}_k(s) + \tilde{F}_k(s)]ds + \sigma_k d\beta_k(s), \\ \tilde{\omega}_k(0) &= \omega_k(0). \end{aligned}$$

While $\tilde{\omega}(s)$ is not a diffusion, it is an adapted Itô process. Notice that

$$\tilde{\omega}_k(t) = e^{-\nu|k|^\alpha t}\omega_k(0) + \int_0^t e^{-\nu|k|^\alpha(t-s)} \tilde{F}_k(s)ds + \int_0^t e^{-\nu|k|^\alpha(t-s)} d\beta_k(s).$$

The first and last term are identical to the first and last terms in the analogous representation of $\omega_k(t)$. Observe that

$$\int_0^t e^{-\nu|k|^\alpha(t-s)}\tilde{F}_k(s)ds = \int_{\frac{t}{2}}^t e^{-\nu|k|^\alpha(t-s)}\tilde{F}_k(s)ds$$
$$= \int_{\frac{t}{2}}^t e^{-\nu|k|^\alpha(t-s)}2e^{-\nu|k|^\alpha(t-s)}F_k(\omega(2s-t))ds.$$

Setting $\tau = 2s - t$, we have

$$\int_0^t e^{-\nu|k|^\alpha(t-s)}\tilde{F}_k(s)ds = \int_0^t e^{-\nu|k|^\alpha(t-\tau)}F_k(\omega(\tau))d\tau.$$

Hence, $\omega(t) = \tilde{\omega}(t)$. Observe that equality holds only at time t and that the distributions on path space are different.

We proceed to show that the auxiliary process, $\tilde{\omega}$, induces a measure on the path space $C([0,t], l^2(\mathbb{Z}^2_*))$ which is equivalent to the measure induced by the Ornstein–Uhlenbeck process. This implies that the transition measures of the auxiliary process and the Ornstein–Uhlenbeck process at time t are equivalent. Since the transition measures of the hyperviscous Navier–Stokes equations and the auxiliary process are equal at time t (by construction), we conclude that the hyperviscous Navier-Stokes process ($\alpha > 2$) and the Ornstein–Uhlenbeck process have equivalent transition densities. This fact leads to a simple proof of unique ergodicity for the Navier–Stokes process; this proof is given in Section 5.3.

We will first make precise the spaces in which we work. We let

$$(\Omega, \mathcal{F}, \mathcal{F}_s, \mu) = (C([0,t], l^2(\mathbb{Z}^2_*)), \mathcal{F}, \mathcal{F}_s, P),$$

where $\mathcal{F}, \mathcal{F}_s$ are the Borel sigma algebra and the filtration generated by finite dimensional distributions up to time s, respectively. P is the measure induced on the path space by the Ornstein–Uhlenbeck process. Note that one can recover the Brownian forcing from the Ornstein–Uhlenbeck process since all the relations are linear and invertible; let $\tilde{T}: \Omega \to \Omega$ be the map that recovers the Brownian paths from the Ornstein–Uhlenbeck process. Next, we let $T: \Omega \to \Omega$ be the identity map. Let $\tilde{\omega}_k^{(N)}$ be defined for all $k \in \mathbb{Z}^2_*$ by

$$d\tilde{\omega}_k^{(N)}(s) = [-|k|^\alpha \tilde{\omega}_k^{(N)}(s) + \tilde{F}_k(s)1_{\{\tau_N > s\}}]ds + \sigma_k d\beta_k(s).$$

Here, $\tau_N = \inf\{s \in [0,t] : \|\omega\| > N, \text{ or } z \notin A_1^s(N,\gamma)\}$, where γ is fixed such that $l+1 < \gamma < l + \alpha/2$. τ_N is a stopping time and by the earlier probabilistic estimates, P-almost surely $\lim_{N\to\infty} \min\{\tau_N, t\} = t$. We note that $\tilde{F}_k^{(N)}(s) = \tilde{F}_k(s) 1_{\{\tau_N > s\}}$ is a bounded Ito process. Let $\tilde{W}_N : \Omega \to \Omega$ be the map, which takes Brownian paths in Ω to $\tilde{\omega}^{(N)}$. This is just the solution map for the SDE for $\omega^{(N)}$. Let $T_N = \tilde{W}_N \circ \tilde{T}$. Define $Q_N = T_N^* P$, the measure induced on Ω by $\tilde{\omega}^{(N)}$. Since $\tau_N \to \infty$ as $N \to \infty$ P-almost surely, for any $A \in \mathcal{F}$, $Q_N(A) \to Q(A)$, where Q is the measure induced on Ω by the process $\tilde{\omega}$.

Girsanov's theorem and a calculation imply that $P \sim Q_N$ for every N. Before doing this calculation, we see that it implies that $Q \ll P$: if $P(A) = 0$, then

$$Q(A) = \lim_{N\to\infty} Q_N(A) = \lim_{N\to\infty} 0 = 0.$$

We have used the assumption that $Q_N(A) \to Q(A)$ for all measurable A and that $P \sim Q_N$. The calculation needed to prove equivalence of the approximations and the Ornstein–Uhlenbeck process is

$$\begin{aligned}
\int_0^t \sum_{k\in\mathbb{Z}_*^2} \frac{|\tilde{F}_k^{(N)}(s)|^2}{|\sigma_k|^2} ds &\leqslant 4 \sum_{k\in\mathbb{Z}_*^2} \left[\sup_{s\in[0,t]} |F_k(\omega(s)) 1_{\{\tau_N > s\}}|^2 \right] \\
&\quad \left[\int_0^t \frac{e^{-2\nu|k|^\alpha (t-s)}}{|\sigma_k|^2} ds \right] \\
&\leqslant \sup_{s\in[0,t]} \sum_{k\in\mathbb{Z}_*^2} \frac{2|F_k(\omega(s)) 1_{\{\tau_N>s\}}|^2}{\nu |k|^\alpha |\sigma_k|^2} \\
&\leqslant \text{ poly } (N) \sum_{k\in\mathbb{Z}_*^2} \frac{|k|^{2l} |k|^2 \log|k|}{|k|^{2\gamma+\alpha}},
\end{aligned}$$

where poly(N) is a fixed polynomial in N. This polynomial bound follows from Lemma 2.2 since $\tau_N > t$ implies that $\omega \in A_1^t(N,\gamma) \cap A_2^t(\|\omega_0\|, \frac{N}{\|\omega_0\|})$. Since $\gamma > l+1$ the last sum converges. Girsanov's theorem allows us to assert that $P \sim Q_N$ for every N. (See Lemma B.1 from Appendix B.)

In order to show that $P \ll Q$, we will need a tail estimate on $P(\tau_N > t)$. We assume that $\mathcal{E}$ is fixed since it is just determined by the initial condition.

$$P(z \notin A_1^t(\mathcal{D},\gamma)) \leqslant \sum_{k\in\mathbb{Z}_*^2} P\left(|z_k| > \frac{\mathcal{D}}{|k|^\gamma}\right).$$

A simple Gaussian tail estimate leads to

$$P\left(|z_k| > \frac{\mathcal{D}}{|k|^\gamma}\right) \leqslant \frac{2|k|^{l+\frac{\alpha}{2}+\gamma}}{\mathcal{D}\sqrt{\pi}} e^{\frac{-\mathcal{D}^2}{2}|k|^{l+\frac{\alpha}{2}-\gamma}} e^{\frac{-\mathcal{D}^2}{2}|k|^{l+\frac{\alpha}{2}-\gamma}}.$$

For $\mathcal{D}$ large enough but fixed,

$$\frac{2|k|^{l+\frac{\alpha}{2}+\gamma}}{\mathcal{D}\sqrt{\pi}} e^{\frac{-\mathcal{D}^2}{2}|k|^{l+\frac{\alpha}{2}-\gamma}}$$

can be made small uniformly in $|k|$.

It is an exercise to show that there is some fixed C such that

$$\sum_{k\in\mathbb{Z}_*^2} e^{\frac{-\mathcal{D}^2}{2}|k|^{l+\frac{\alpha}{2}-\gamma}} \leqslant Ce^{-\frac{\mathcal{D}^2}{2}}$$

for $\mathcal{D}$ sufficiently large.

Assume $\|\omega_0\| < \mathcal{E}$. By Lemma 3.1, we have that

$$P(\omega \notin A_2^t(\mathcal{E},\eta)) \leqslant \exp\left(-\frac{\nu}{\sigma_{\max}^2}[(\eta-1)\mathcal{E} - \mathcal{E}_1 t]\right)$$

for η sufficiently large. By the definition of τ_N and these exponential estimates, we see that there is a positive constant c such that $P(\tau_N > t) < e^{-cN}$. We will use this bound to prove that $P \ll Q$. By the Lemma A.2, it suffices to show that $H(P|Q_N)$ is uniformly bounded: $\sup_N \int log(\frac{dP}{dQ_N})dP < M < \infty$. Since the Radon–Nikodym derivative dP/dQ_N is a local exponential martingale, we need only show that

$$\int \left[\int_0^t \sum_{k\in\mathbb{Z}_*^2} \frac{|\tilde{F}_k(s)|^2}{|\sigma_k|^2} ds\right] dP < \infty.$$

In order to show this we apply Fatou's lemma and a simple stopping time argument. As usual, we denote by E the expectation with respect to P.

$$
\begin{aligned}
&\int \left[\int_0^t \sum_{k\in\mathbb{Z}_*^2} \frac{|\tilde{F}_k(s)|^2}{|\sigma_k|^2} ds\right] dP \leqslant \lim_{N\to\infty} E\left[\sum_{k\in\mathbb{Z}_*^2}\int_0^t \frac{|\tilde{F}_k^{(N)}(s)|^2}{|\sigma_k|^2} ds\right]\\
&= \lim_{N\to\infty}\left\{\sum_{k\in\mathbb{Z}_*^2}\left(E\left[\int_0^t \frac{|\tilde{F}_k(s)|^2}{|\sigma_k|^2} ds 1_{\tau_N>t}\right] + E\left[\int_0^t \frac{|\tilde{F}_k^{(N)}(s)|^2}{|\sigma_k|^2} ds 1_{\tau_N\leqslant t}\right]\right)\right\}\\
&\leqslant \lim_{N\to\infty}\sum_{k\in\mathbb{Z}_*^2} E\left[\int_0^t \frac{|\tilde{F}_k(s)|^2}{|\sigma_k|^2} ds 1_{\tau_N>t}\right]\\
&\quad + \lim_{N\to\infty}\left[poly(N)e^{-cN}\sum_{k\in\mathbb{Z}_*^2}\frac{|k|^{2l}|k|^2\log|k|}{|k|^{2\gamma+\alpha}}\right]\\
&= \lim_{N\to\infty}\sum_{k\in\mathbb{Z}_*^2} E\left[\int_0^t \frac{|\tilde{F}_k(s)|^2}{|\sigma_k|^2} ds 1_{\tau_N>t}\right]\\
&= \lim_{N\to\infty}\sum_{l=1}^{N}\sum_{k\in\mathbb{Z}_*^2} E\left[\int_0^t \frac{|\tilde{F}_k(s)|^2}{|\sigma_k|^2} ds 1_{\tau_l>t\geqslant\tau_{l-1}}\right]\\
&\leqslant \left[\sum_{k\in\mathbb{Z}_*^2}\frac{|k|^{2l}|k|^2\log|k|}{|k|^{2\gamma+\alpha}}\right]\lim_{N\to\infty}\sum_{l=1}^{N} poly(l)e^{-c(l-1)} < \infty.
\end{aligned}
$$

This completes the proof of Theorem 2.

5.3. Invariant Measures and Ergodicity

In this section, we show that hyperviscous SNS has a unique invariant measure, ν^{ω}. This measure is equivalent to the Ornstein–Uhlenbeck invariant measure, ν^{z}. By Section 5, we know that if $\alpha > 2$ then $P_t(x,\cdot) \sim Q_t(x,\cdot)$, where $P_t(x,\cdot)$, and $Q_t(x,\cdot)$ are the transition kernels starting at x for the SNS process and Ornstein–Uhlenbeck process, respectively, and $\sim$ denotes equivalence of measures. Since $Q_t(x,\cdot) \sim Q_t(y,\cdot)$ for all $x, y \in l^2(\mathbb{Z}_*^2)$ (simple to check since the semigroup of Ornstein–Uhlenbeck is sufficiently contractive) $Q_t(y,\cdot) \sim P_t(x,\cdot)$ for all $x, y \in l^2(\mathbb{Z}_*^2)$. Invariance of the measures can be stated as

$$\nu^\omega(A) = \int_{l^2(\mathbb{Z}_*^2)} P_t(x,A)\nu^\omega(dx),$$
$$\nu^z(A) = \int_{l^2(\mathbb{Z}_*^2)} Q_t(x,A)\nu^z(dx).$$

Existence of such a measure for the Ornstein–Uhlenbeck process is immediate as it can be constructed explicitly. For the hyperviscous SNS, existence follows from tightness arguments that have become standard.[2,5]

Suppose $\nu^\omega(A) > 0$ and let $\epsilon_n = \frac{1}{n}$. Define $B_n = \{x \in l^2(\mathbb{Z}_*^2) : Q_t(x,A) > \epsilon_n P_t(x,A) > 0\}$. In order to avoid any confusion, we remark that $P_t(x,A) = 0$ if and only if $Q_t(x,A) = 0$ by our remarks on equivalence. Note that

$$\nu^z(A) = \int_{l^2(\mathbb{Z}_*^2)} Q_t(x,A)\nu^z(dx) \geqslant \int_{B_n} Q_t(x,A)\nu^z(dx) \geqslant \epsilon_n \int_{B_n} P_t(x,A)\nu^z(dx).$$

If $\nu^z(l^2(\mathbb{Z}_*^2)/\bigcup_n B_n) = 0$, then $\nu^z(A) > 0$ since $\nu^z(B_n) > 0$ for some n and $P_t(x,A) > 0$ for all $x \in B_n$. On the other hand, let $\mathcal{L} = (l^2(\mathbb{Z}_*^2)/\bigcup_n B_n)$ and suppose $\nu^z(\mathcal{L}) > 0$. By the previous remark, for every $x \in \mathcal{L}$, $Q_t(x,A) = 0$; this implies $P_t(x,A) = 0$. $P_t(x,\cdot) \sim P_t(y,\cdot)$ for all $y \in l^2(\mathbb{Z}_*^2)$, thus $P_t(y,A) = 0$ for all $y \in l^2(\mathbb{Z}_*^2)$; but, this is impossible since $\nu^\omega(A) > 0$. This implies $\nu^z(B_n) > 0$ for some n, so $\nu^\omega \ll \nu^z$. Similarly, we can show $\nu^z \ll \nu^\omega$. Since $\nu^z \sim \nu^\omega$ for any two ω invariant ergodic measures, we know that these two measures are equivalent; therefore, they must be the same measure by a standard ergodic theory argument. ∎

It is important to realize that not every infinite dimensional Ornstein–Uhlenbeck process has transition densities which are absolutely continuous for different initial conditions. This is true in our setting because the semigroup is sufficiently contractive and the forcing decays slowly enough.

6. CONCLUDING REMARKS

In this paper we have proven three theorems. They demonstrate different ways to interpret the phrase "the small scales are similar". The results were given in increasing strength. The first is a weak convergence type of result. It states that the rescaled modes of the Navier–Stokes equations converge to those of a naturally associated Ornstein–Uhlenbeck process as the scales become smaller or the wave number increase. The second theorem states that the hyperviscous Navier–Stokes equations ($\alpha > 2$) and its associated Ornstein–Uhlenbeck process induce equivalent measure on phase space at any fixed time t. In other words, the Markov transition kernels of the two processes at a fixed time are equivalent. This gives a simple proof of unique ergodicity for the hyperviscous Navier–Stokes equations. The third theorem states that the hyperviscous Navier–Stokes equations ($\alpha > 4$) and its associated Ornstein–Uhlenbeck process induce equivalent measure on path space. As a result we see that the hyperviscous Navier–Stokes equation has a unique invariant measure.

APPENDIX A. A TECHNICAL LEMMA

In this section, we prove the technical Lemma 2.2. It differs little from the arguments of [11]. By Cauchy–Schwartz,

$$|F(\omega)_k(t)| \leqslant G(\omega)_k(t) = \sum_{l_1+l_2=k} |\omega_{l_1}(t)||\omega_{l_2}(t)|\Big|\frac{(k, l_2^\perp)}{(l_2, l_2)}\Big|.$$

We estimate $\sup_{s\leqslant t} G(\omega)_k(s)$ given that $\omega \in A_1^t(\mathcal{D}, \gamma) \cap A_2^t(\mathcal{E}, \eta)$. We begin by breaking the above sum into three parts:

$$\Sigma_1 = \sum_{|l_2|\leqslant \frac{|k|}{2}}, \qquad \Sigma_2 = \sum_{2|k|\geqslant |l_2| > \frac{|k|}{2}}, \qquad \Sigma_3 = \sum_{|l_2|>2|k|}.$$

A.1. To estimate Σ_1, we note that $\big|\frac{(k,l_2^\perp)}{(l_2,l_2)}\big| \leqslant \frac{|k|}{|l_2|}$ and $|\omega_{l_1}| \leqslant \frac{2^\gamma \mathcal{D}}{|k|^\gamma}$. Hence

$$\Sigma_1 \leqslant \frac{2^\gamma \mathcal{D}}{|k|^{\gamma-1}} \sum_{|l_2|\leqslant \frac{|k|}{2}} \frac{|\omega_{l_2}|}{|l_2|} \leqslant \frac{2^\gamma \mathcal{D}}{|k|^{\gamma-1}} \sqrt{\sum_{l_2} |\omega_{l_2}|^2} \sqrt{\sum_{|l_2|\leqslant \frac{|k|}{2}} \frac{1}{|l_2|^2}}$$
$$\leqslant \frac{2^\gamma \mathcal{D}}{|k|^\gamma} \sqrt{\eta \mathcal{E}} M|k|\sqrt{\ln |k|},$$

where M is a constant arising from the second summation and does not depend on any of the parameters.

A.2. To estimate Σ_2, we note that since $\frac{|k|}{2} < |l_2| \leqslant 2|k|$, the inequalities $\left|\frac{(k,l_2^\perp)}{(l_2,l_2)}\right| \leqslant 2$ and $|\omega_{l_2}| \leqslant \frac{2^\gamma \mathcal{D}}{|k|^\gamma}$ hold. Thus,

$$\Sigma_2 \leqslant \frac{2^{\gamma+1}\mathcal{D}}{|k|^\gamma} \sum_{|l_1|\leqslant 3|k|} |\omega_{l_1}| \leqslant \frac{2^{\gamma+1}\mathcal{D}}{|k|^\gamma}\sqrt{\eta\mathcal{E}}\sqrt{\sum_{|l_1|\leqslant 3|k|} 1} \leqslant \frac{2^{\gamma+1}\mathcal{D}}{|k|^\gamma}\sqrt{\eta\mathcal{E}}(6|k|+1).$$

A.3. Estimating Σ_3, we find

$$\begin{aligned}\Sigma_3 \leqslant |k| \sum_{|l_2|>2|k|} |\omega_{l_1}| \frac{|\omega_{l_2}|}{|l_2|} &\leqslant |k|\sqrt{\eta\mathcal{E}}\sqrt{\sum_{|l_2|>2|k|} \frac{|\omega_{l_2}|^2}{|l_2|}} \\ &\leqslant |k|\sqrt{\eta\mathcal{E}}\mathcal{D}\sqrt{\sum_{|l_2|>2|k|} \frac{1}{|l_2|^{2(\gamma+1)}}} \leqslant |k|\sqrt{\eta\mathcal{E}}\bar{M}(\gamma)\frac{\mathcal{D}}{|k|^\gamma},\end{aligned}$$

where $\bar{M}(\gamma)$ depends only on γ through the estimate on the last sum.

Adding the above estimates for the three sums, we see that

$$\sup_{s\leqslant t} G(\omega)_k(s) \leqslant \sqrt{\eta\mathcal{E}}\frac{\mathcal{D}}{|k|^\gamma}(|k|\sqrt{\ln|k|})C(\gamma),$$

which proves the lemma.

APPENDIX B. COMPARISON OF MEASURES ON PATH SPACE

Suppose that we have stochastic processes $X_i(t)$, $i = 1, 2$ on the path space $C([0,T],\mathbb{X})$, where $\mathbb{X}$ is some separable Hilbert space and $T \in (0,\infty]$. Furthermore, assume that X_i satisfies the equation

$$\begin{aligned} dX_i(t) &= f_i(t, X_i[0,t])dt + g\,dW(t), \quad t \in [0,T], \\ X_i(0) &= x_0. \end{aligned} \tag{14}$$

For fixed t, the functions f_1 and f_2 map the space $C_{[0,t]} = C([0,t],\mathbb{X})$ to $\mathbb{X}$. By $X[0,t]$ we mean the segment of the trajectory on $[0,t]$. $W(t)$ is a cylindrical Brownian motion over a separable Hilbert space $\mathbb{Y}$ and g is a fixed invertible Hilbert–Schmidt operator from $\mathbb{Y} \to \mathbb{X}$. For any $\mathcal{B} \subset C_{[0,T]}$, define measures $P^{(i)}_{[0,T]}(\,\cdot\,;\mathcal{B})$ on the path space as

$$P^{(i)}_{[0,T]}(A;\mathcal{B}) = P\{X_i[0,T] \in A \cap \mathcal{B}\}, \quad \text{for } A \subset C_{[0,T]}.$$

Define also $D(t, \cdot) = f_1(t, \cdot) - f_2(t, \cdot)$.

In this setting, we have the following result which is a variation on Lemma B.1 from ref. 8 and follows quickly from Girsanov's Theorem.

Lemma B.1. Assume there exists a constant $D_* \in (0, \infty)$ such that

$$\exp\left\{\frac{1}{2}\int_0^T \left|g^{-1}D\big(t, X_i[0,t]\big)\right|_{\mathbb{Y}}^2 dt\right\} \mathbf{1}_{\mathcal{B}}(X_i[0,t]) < D_* \tag{15}$$

almost surely for $i = 1, 2$. Then the measures $P^{(1)}_{[0,T]}(\cdot\,;\mathcal{B})$ and $P^{(2)}_{[0,T]}(\cdot\,;\mathcal{B})$ are equivalent.

Proof. Define the auxiliary SDEs

$$dY_i(t) = f_i\big(t, Y_i[0,t]\big)\mathbf{1}_{\mathcal{B}(t)}(Y_i[0,t])dt + g\,dW(t),$$

where $\mathcal{B}(t) = \{x \in C_{[0,t]} : \exists \bar{x} \in \mathcal{B}$ such that $x(s) = \bar{x}(s)$ for $s \in [0,t]\}$. Solutions $Y_i(t)$ to these equations can be constructed as

$$Y_i(t) = X_i(t)\mathbf{1}_{\{t \le \tau\}} + [gW(t) - gW(\tau) + X_i(\tau)]\mathbf{1}_{\{t > \tau\}}.$$

Here $\tau = \inf\{s > 0 : X_i[0,s] \notin \mathcal{B}(s)\}$.

Denote $D_{\mathcal{B}}(t,x) = [f_1(t,x) - f_2(t,x)]\mathbf{1}_{\mathcal{B}(t)}(x)$. The assumption on D in Eq. (15) and the definition of $\mathcal{B}(t)$ imply that

$$\exp\left\{\frac{1}{2}\int_0^T \left|g^{-1}D_{\mathcal{B}}\big(t, X[0,t]\big)\right|_{\mathbb{Y}}^2 dt\right\} < D_* \quad \text{a.s.}$$

under both measures $P^{(i)}_{Y[0,t]}$ defining solutions to auxiliary equation with $i = 1$ and $i = 2$. Hence, Novikov's condition is satisfied for the difference in the drifts of the auxiliary equations. Girsanov's theorem implies that

$$\frac{dP^{(1)}_{Y[0,t]}}{dP^{(2)}_{Y[0,t]}}(x) = \mathcal{E}(x),$$

where the Radon–Nikodym derivative evaluated at a trajectory x is defined by the stochastic exponential

$$\mathcal{E}(x) = \exp\left\{\int_0^T \left\langle g^{-1}D_{\mathcal{B}}(s, x[0,s]), dW(s)\right\rangle_{\mathbb{Y}} - \frac{1}{2}\int_0^T |g^{-1}D_{\mathcal{B}}(s, x[0,s])|_{\mathbb{Y}}^2 ds\right\}.$$

Note that restrictions of measures $P^{(i)}_{Y[0,t]}$ on the set $\mathcal{B}$ coincide with $P^{(i)}_{[0,t]}(\cdot\ ;\mathcal{B})$. This proves that $P^{(1)}_{[0,t]}(\cdot\ ,\mathcal{B})$ is absolutely continuous with respect to $P^{(2)}_{[0,t]}(\cdot\ ;\mathcal{B})$. The reverse relation follows by symmetry and the proof is complete. ∎

APPENDIX C. RELATIVE ENTROPY AND EQUIVALENCE OF MEASURES

Lemma C.1 provides a sufficient condition for showing the absolute continuity of a fixed measure with respect to measure arising as the limit of certain approximating measures.

Lemma C.1. Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and let $(W, \mathcal{W})$ be a measure space. Assume W is a Polish space and $\mathcal{W}$ is the Borel sigma algebra. Let $T:\Omega\to W$ and $T_n:\Omega\to W$ $(n=1,2,\cdots)$ be measurable transformations. Let $P=T^*\mu$ and $Q_n=T_n^*\mu$ be the push–forward measures on W induced by the respective transformations. Assume that there is a probability measure Q on W such that for any measurable $A\in\mathcal{W}$, $Q_n(A)\to Q(A)$. If $P\sim Q_n$ and $\limsup_{n\to\infty}|\int\frac{dP}{dQ_n}log\frac{dP}{dQ_n}dQ_n|<M<\infty$, then $P\ll Q$.

Proof. Denote by $H(\mu|\nu)=\int\frac{d\mu}{d\nu}\log\frac{d\mu}{d\nu}d\nu$ the relative entropy of the probability measure μ with respect to ν (when it exists). We begin by proving a basic inequality. If μ and ν are mutually absolutely continuous, $f\in L^1(\mu)$ and $H(\mu|\nu)<\infty$, then

$$\int f\,d\mu\leqslant H(\mu|\nu)+\log\left(\int e^f\,d\nu\right).$$

This inequality follows from the simple calculation:

$$\begin{aligned}\int f\,d\mu-\log\left(\int e^f\,d\nu\right) &= \int f\,d\mu-\log\left(\int e^f\frac{d\nu}{d\mu}d\mu\right)\\ &\leqslant \int f\,d\mu-\int\log\left(e^f\frac{d\nu}{d\mu}\right)d\mu\\ &= \int\log\frac{d\mu}{d\nu}d\mu=H(\mu|\nu)\end{aligned}$$

In particular, for any $c>0$ the inequality becomes

$$\int f\,d\mu\leqslant\frac{1}{c}H(\mu|\nu)+\frac{1}{c}\log\left(\int e^{cf}\,d\nu\right).$$

Letting $f = \chi_A$, the characteristic function of a set $A \in \mathcal{W}$, this inequality becomes

$$P(A) \leqslant \frac{1}{c} H(P|Q_n) + \frac{1}{c} \log((e^c - 1)Q_n(A) + 1).$$

Fix $c > 0$. If $Q(A) = 0$, then $Q_n(A) \to 0$ by assumption. Since lim sup $H(P|Q_n) < M < \infty$, as $n \to \infty$ the right hand side is bounded by $2M/c$. $P(A) = 0$ since c is arbitrary. Thus, $P \ll Q$. ▮

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